9/17/25

Last Time

$A = m \times p$ matrix

$B = p \times n$ matrix

$\| $

$[\vec{b}_1, \dots \vec{b}_n]$

$AB = [A\vec{b}_1 \dots A\vec{b}_n]$

Example:

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem:

If $R: \mathbb{R}^p \to \mathbb{R}^m$, $S: \mathbb{R}^n \to \mathbb{R}^p$ are linear transformations then $R \circ S$ is linear and $\underbrace{[R \circ S]}_{AB} = \underbrace{[R]}_{A}\underbrace{[S]}_{B}$

OTOH $(R \circ S)(\vec{x}) = (AB)(\vec{x})$

OTOH $R(S(\vec{x})) = R(B\vec{x}) = A(B\vec{x})$

so $(AB)(\vec{x}) = A(B\vec{x})$

$\boxed{\text{e.g.}}$ Why is $AI = A$?

proof 1: Let $R(\vec{y}) = A\vec{y}$, $id(\vec{x}) = \vec{x}$

then $[R \circ id] = [R][id] = AI$

and $R \circ id = R(\vec{x})$

So $[R \circ id] = [R] = A$

i.e. $A = AI$

proof 2:

$$AI = A[\vec{e}_1 \ldots \vec{e}_n]$$
$$= [A\vec{e}_1 \ldots A\vec{e}_n]$$
$$= \begin{bmatrix} 1^{st}\text{ col of } A & \cdots & n^{th}\text{ col of } A \end{bmatrix} = A$$

$\boxed{\text{e.g.}}$ $A \cdot (B + C) = A \cdot \left([\vec{b}_1 \ldots \vec{b}_n] + [\vec{c}_1 \ldots \vec{c}_n] \right)$ write B+C column wise

$= A[\vec{b}_1 + \vec{c}_1, \ldots, \vec{b}_n + \vec{c}_n]$ definition of matrix addition

$= [A(\vec{b}_1 + \vec{c}_1) \ldots A(\vec{b}_n + \vec{c}_n)]$ definition of matrix multiplication

$= [A\vec{b}_1 + A\vec{c}_1 \ldots A\vec{b}_n + A\vec{c}_n]$ distributive Law for matrix times vector

$= [A\vec{b}_1 \ldots A\vec{b}_n] + [A\vec{c}_1 \ldots A\vec{c}_n]$ definition of matrix addition

$= A[\vec{b}_1 \ldots \vec{b}_n] + A[\vec{c}_1 \ldots \vec{c}_n]$ definition of matrix multiplication

$= AB + AC$

Prop - Matrix multiplication is associative

Let: A be $m \times p$ matrix,
B be $p \times l$ matrix,
C be $l \times n$ matrix

then $AB(C) = A(BC)$

Let $T(\vec{x}) = C\vec{x}$, $S(\vec{y}) = B(\vec{y})$, $R(\vec{z}) = A\vec{z}$

Then - $(R \circ S) \circ T(\vec{x}) = (R \circ S)(T(\vec{x})) = R(S(T(\vec{x})))$

OTOH!

$R \circ (S \circ T)(\vec{x}) = R(S \circ T(\vec{x})) = R(S(T(\vec{x})))$

So

$R \circ (S \circ T) = (R \circ S) \circ T$

$\rightarrow [R \circ (S \circ T)] = [(R \circ S) \circ T]$

$\rightarrow [R][S \circ T] = [R \circ S][T]$

$\rightarrow [R]([S][T]) = ([R][S])[T]$

$\rightarrow A(BC) = (AB)C$

$\blacksquare$